How Should Public Debt Management Institutions Develop Medium-Term Issuance Strategies?

An optimal debt portfolio model

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Abstract

Public debt management institutions face challenges in developing medium-term debt management strategies. The main objective of public debt management is the funding of government borrowing needs, with two goals in mind: minimizing costs while keeping risks at an acceptable level. However, these two goals are contradictory as lower costs can usually be achieved only by taking higher risks. Therefore, formalizing a comprehensive public debt management strategy and analyzing its effects is crucial. Our purpose is to find Pareto-optimal composition of borrowing that satisfy these goals with the help of stochastic financial modeling. We propose an optimal portfolio model, where we apply a multi-objective optimization scheme to construct composition of borrowing that can be considered by the debt manager as feasible alternatives. We apply our model on the Hungarian government debt portfolio, which is relatively complex by having a significant share of foreign currency debt and special retail debt securities as well. Results show that our model is capable of providing a complex analysis of desired issuance strategies and borrowing compositions based on the debt manager's preferences regarding all important aspects of debt management, especially risk management. The model can be modified to satisfy the unique needs of different DMOs in creating medium-term debt management strategies.

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Results published in this paper, including but not limited to macroeconomic forecasts and the composition of the Hungarian government debt, do not represent the official views of the Government Debt Management Agency Pte. Ltd. regarding debt financing. The only purpose of disclosing such results is to illustrate the features and possibilities of the optimal debt portfolio model. The authors give no warranty and make no representation as to the accuracy, reliability, timeliness or other features of any data contained in this paper or data obtained from using the model. All models must be scientifically validated by the user for the strategy for which it is to be used, and for the most appropriate and safe application of models, scientific and expert interpretation and adequate advice is required.

1. Introduction

In this paper, we demonstrate the construction of an optimal debt portfolio model with the purpose of supporting the development of a medium-term issuance strategy for the Government Debt Management Agency Pte. Ltd. being the Hungarian DMO. Public debt management institutions face the challenge of funding government borrowing needs at a minimal cost subject to an acceptable risk level. These two goals are in most cases contradictory as lower costs can usually be achieved only by taking higher risks. The constraint of keeping risk levels "acceptable" is also subject to interpretation. Therefore, formalizing a comprehensive strategy and analyzing its effects is crucial. The purpose of the model is to find Pareto-optimal composition of borrowing that satisfy these goals with the help of stochastic financial modeling.

An optimal debt portfolio model can help the Hungarian Government Debt Management Agency Pte. Ltd. in creating a medium term (5-year) debt management strategy. Several countries have developed stochastic optimal portfolio models for debt management purposes. Notable examples include Canada (Bolder and Deeley 2011), Italy (Adamo et al. 2004; Consiglio and Staino 2012), Sweden (Bergström, Holmlund and Lindberg 2002), Turkey (Balibek and Memis 2012) and the United Kingdom (Pick and Anthony 2006).

Compared to other countries, we face relatively unique challenges concerning the attributes of the Hungarian government debt portfolio. It is relatively complex, containing both fixed and floating rate debt, significant foreign currency (FX) debt denominated in multiple currencies (but swapped to Euros), and a sizeable (and growing) share of retail debt, including inflation-linked securities.

This paper is a combination and extension of two of our previous papers. One of them (Bebes, Tran and Bebesi 2018b) focuses on forecasting, while the other (Bebes, Tran and Bebesi 2018a) focuses on optimization. This standalone study gives a complete picture of the Hungarian optimal debt portfolio model and goes beyond the scope of our previous papers in several ways.

First, we treat the sizeable Hungarian retail debt as an external factor regarding optimization. This is closer to reality, as the Government Debt Management Agency cannot refuse retail buyers. However, we use a more complex estimation of retail demand than before, incorporating it into the forecasting model similar to macro variables. Next, the issuance algorithm is overhauled to include the balance of the Treasury Single Account as an important issue. The calculation of liquidity costs is also changed to allow excess issuances to have temporal spillover effects by elevating future yields. Furthermore, this paper includes discussion on determining the optimal resource allocation for the optimization algorithm. Finally, we updated all results to reflect these changes, and the forecasting horizon has been changed to December 2019-December 2024.

The structure of this paper is as follows: Section 2 gives an overview of the optimal debt portfolio model. Section 3 introduces the data used for the model. Section 4 deals with the specification of the used Markov regime switching state-space model and gives an overview of the process of parameter estimation. Section 5 deals with issues regarding the estimation of costs and risks. Section 6 discusses the optimization and simulation process. Section 7 analyzes the results of the optimization. Section 8 concludes.

2. Overview of the model

In this section, we give an overview of the various phases and steps of the Hungarian optimal debt portfolio model. We also explain the modeling terms we use in later sections of the paper.

The first phase of the model is the construction of a Markov regime switching state-space model. We use this model to simultaneously forecast the Hungarian and the Euro area term structures of interest rates as well as some macroeconomic and financial factors, namely the Hungarian inflation (CPI) rates, the Hungarian credit default swap (CDS) curve, the EUR/HUF exchange rate and the demand for Hungarian retail debt instruments. The two yield curves and the macro factors are adequate for modeling the future cash-flows of both the outstanding government debt securities and the instruments to be issued in the future by the Hungarian state.

Modeling and forecasting the term structure of interest rates (yield curve) is a well-researched topic. The most popular class of models used in empirical applications are variants of the Nelson-Siegel (1987) model. A dynamic extension of the original static model was proposed by Diebold and Rudebusch (2013). We model the yield curve using a rotated dynamic Nelson-Siegel Model following the work of Nyholm (2015).

State-space models are useful for modeling dynamic linear systems driven by unobservable parameters. These parameters can be estimated using the Kalman filter (1960). In practice, however, economic applications often exhibit non-linearity. Hamilton (1989) was the first to apply Markov regime switching models to economic problems with the purpose of dealing with time varying parameters. The unobservable parameters of Markov regime switching models can be estimated using an extension of the Kalman filter by Kim (1994). Our work follows Kim and Nelson (1999) in using Gibbs sampling for parameter estimation and the Kim filter for determining the factor values.

Using a regime switching model is easily justified by economic considerations. Economic cycles are prevalent and can be separated into longer periods of growth (expansion) and shorter crisis periods of contraction (recession). These two distinct empirical situations allow for a meaningful application of time varying parameters. The steady state values (long-run expectations) determining the factors, the transition matrix and the error terms are all different based on the given regime. The heteroscedasticity of the error terms of the transition equation driving the factors can be handled using a different covariance matrix for every state.

Using a two-economy model with additional macro factors is also necessary as the Hungarian debt portfolio includes foreign currency denominated bonds as well as inflation-linked securities. Although the portfolio includes securities denominated in several currencies, the entire foreign currency exposure is only Euro denominated due to cross-currency swaps.

We can also model the dynamics between a small open economy (Hungary) and a large economy (the Euro area). Our model allows factor dynamics to be specified in a way that the Euro area factors can influence the Hungarian factors but not vice versa.

In the second phase of the model we generate a series of random numbers based on the estimated stochastic distributions. We define a *trajectory* as the realized time series of model factors. Usually, we simulate $1{,}000$ trajectories.

The third phase of the model is optimization. In this phase, we search for Pareto-optimal borrowing compositions. *A borrowing composition* is a weighting of the model instruments, summing to 1, while aiming for the theoretical composition of the portfolio, as time $t \to \infty$. Model instruments include

Hungarian Government Bonds and Treasury Bills as well as retail instruments and foreign currency bonds, as described in Section 6.1.

Starting from a random array of borrowing compositions, the optimization algorithm (Sun, Zhu and Cai 2017) improves and refines through several *iterations*, finding the corner compositions first and filling the optimization space afterwards. *The issuance algorithm* runs for every trajectory, borrowing composition and iteration with a monthly frequency. The cost and risk metrics are calculated based on the simulated issuances. We generally use 25,000 borrowing compositions and 200 iterations. The rationale for this ratio is discussed in Section 6.4. With a simulation horizon of over 60 months, the issuance algorithm runs over 300 billion times[3].

3. Data

In this section, we give an overview of the macro-financial and yield curve data used for fitting the Markov regime switching model. We also address the challenges regarding the construction of yield curves and introduce our prior assumptions regarding the model. We use data with monthly frequency from October 2008 to December 2019 for a total of $n = 135$ observations for each time series. The limitation is due to the lack of earlier Hungarian CDS data. This gives more than 10 years of historical data and most importantly, includes the 2008 crisis and the following recession period, giving merit to using a regime switching model.

Our data sources include Refinitiv for CDS curve and Euro area yield curve points, the Hungarian Central Statistical Office for Hungarian CPI data and the Hungarian National Bank for EUR/HUF exchange rate time series.

To obtain Hungarian yield curve data, we use the secondary market bid-ask quotes of HUF denominated Hungarian government bonds, as price quotation provided by primary dealers are considered to be actual market prices, the fixing process of which is regulated by the Hungarian Government Debt Management Agency.

We use the Fama-Bliss (1987) method to transform price quotes into yield curve points upon which we can fit the Nelson-Siegel model. This non-parametric method allows us to obtain zero-coupon yield curve points from bond price quotes. Then the Nelson-Siegel model can be fitted on the unsmoothed Fama-Bliss yields.

We impose additional restrictions compared to the original Fama-Bliss (1987) algorithm, using only fixed interest rate bonds with a maturity over 1 year as well as zero-coupon Discount T-bills. Our implementation of the Fama-Bliss (1987) algorithm can be summarized in four steps:

1. We determine whether the Yield-to-Maturity (YTM) of a given security is between the moving average of the previous 3 and the next 3 (based on time to maturity) government securities' yields.
2. We examine the unsmoothed Fama-Bliss forward yields. We exclude a government security from the analysis if the forward rate corresponding to its maturity shows a change greater than 0.2 percentage points in the opposite direction. In this step we calculate the forward rates using only government securities passing step 1.

[3] We use CPU+GPU hybrid computing. The issuance algorithm and the optimization algorithm are calculated by the CPU. Costs and risks are calculated by the GPU parallel to the issuance algorithm. The optimization takes 2-3 hours depending on the number of model instruments on an AMD Ryzen Threadripper 2990WX CPU and an Nvidia 1070Ti GPU.

3. We calculate the forward yield curve once more from the remaining government securities. In the third step we determine whether a previously excluded security can be included in the selection set. The inclusion criteria is similar as in step 1, but instead of YTM the forward yields are calculated. An instrument is to be considered in the analysis if the forward rate corresponding to its maturity is between the two moving averages or the difference from one of them is not larger than 0.2 percentage points.
4. In step 4, step 2 is repeated for the selection set after step 3.

In the end, we get a forward yield curve that fits the two smoothness criteria defined in steps 2 and 3. Then the forward yield curve can be transformed into a zero-coupon yield curve.

Fitting a Markov regime switching model requires us to give an initial estimation of the transition probability matrix (Eq. 5). We set the starting value to: $P_{prior} = \begin{pmatrix} 0.98 & 0.02 \\ 0.18 & 0.80 \end{pmatrix}$. This means that the probability of going from the first regime to the second would be 2%, while exiting from the second regime would be 18%. Thus, the first regime is expected to last 4.2 years, whereas the expected duration of the second regime is less than half a year. The prior steady state values of the factors were determined using P_{prior} and incorporate the effects of the current historically low interest rate period.

The Hungarian CPI expectation in the first regime is slightly above the inflation target of the Hungarian Central Bank ($cpi_{prior,HUN} = 3.5\%$), and is 0.6 percentage point higher in the second regime. We expect the net retail demand in a month in the first regime to be 75 billion HUF. Regarding the HUF yield curve, we expect at least a 50bp increase in the short rate compared to 2019 December, and a 3.5% level for the long end of the yield curve. Our forecast for the yield curve also includes a flattening slope and a rising level in case of a crisis.

Finally, we expect the EUR/HUF exchange rate to slightly rise during the first regime and rise rapidly during the second regime, meaning a comparatively weaker HUF and stronger Euro in a crisis situation.

4. The markov regime switching state-space model

In this section, we construct a Markov regime switching state-space model that can be used to model the entirety of the Hungarian government debt portfolio and explore the relationship between the examined factors in a coherent and realistic way.

To analyze the cost and risk features arising from issuing different kinds of instruments one needs to assess all the kinds of macro-economic factors driving the prices of these underlying instruments. When measuring these factors one also needs to pay attention to the interactions between them. To do that we propose a Markov regime switching state-space model, where we also include different regimes to account for different economic cycles.

4.1 The model

A 3-factor Nelson-Siegel model is used to estimate the HUF yield curve. We assume that the Hungarian Euro yield curve at a specific maturity τ is the sum of the risk-free Euro yield curve and the Hungarian Credit Default Swap rate at maturity τ. A 3-factor Nelson-Siegel model is used to estimate the risk-free Euro yield curve.

Following Nyholm (2015) we rotate the level and slope factors of the Nelson-Siegel model to get a more transparent structure for our factors. Thus our level parameter reflects the level of the short-term rate, and the slope factor is positive in case of a normal (upward sloping) yield curve.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ be the rotation matrix and $E_t(\tau) = \begin{bmatrix} 1 & \frac{1-e^{-\tau/\lambda_t}}{\tau/\lambda_t} \end{bmatrix}$ be the factor loadings for the level and slope parameters, thus $E_t(\tau)A^{-1} = \begin{bmatrix} 1 & 1-\frac{1-e^{-\tau/\lambda_t}}{\tau/\lambda_t} \end{bmatrix}$ denotes the factor loadings for the rotated Nelson-Siegel factors.

In accordance with our assumptions the HUF ($y_{t,H}(\tau)$) and the risk-free Euro ($y_{t,E}(\tau)$) yield curve at time t and maturity τ are given by

$$y_{t,H}(\tau) = L_{t,H} + S_{t,H}\left(1-\frac{1-e^{-\tau/\lambda_{t,H}}}{\tau/\lambda_{t,H}}\right) + C_{t,H}\left(\frac{1-e^{-\tau/\lambda_{t,H}}}{\tau/\lambda_{t,H}} - e^{-\tau/\lambda_{t,H}}\right) + \varepsilon_{\tau,H}, \tag{1}$$

$$y_{t,E}(\tau) = L_{t,E} + S_{t,E}\left(1-\frac{1-e^{-\tau/\lambda_{t,E}}}{\tau/\lambda_{t,E}}\right) + C_{t,E}\left(\frac{1-e^{-\tau/\lambda_{t,E}}}{\tau/\lambda_{t,E}} - e^{-\tau/\lambda_{t,E}}\right) + \varepsilon_{\tau,E}, \tag{2}$$

where $\varepsilon_{\tau,H} \sim N(0, \sigma^2_{\tau,H})$ and $\varepsilon_{\tau,E} \sim N(0, \sigma^2_{\tau,E})$, $t = 1, \dots, n$. One can easily see that $\lim_{\tau\to 0} y_{t,k}(\tau) = L_{t,k}$ and $\lim_{\tau\to\infty} y_{t,k}(\tau) = L_{t,k} + S_{t,k}$ for $k \in \{H, E\}$, where now L denotes the short rate, and S the slope. The curvature factor remains unchanged.

To avoid negative CDS spreads we model the natural logarithm of the CDS curve with a 2-Factor Nelson-Siegel model. Thus, the Hungarian CDS curve ($y_{t,CDS}(\tau)$) at time t and maturity τ is given by

$$\log\left(y_{t,CDS}(\tau)\right) = L_{t,CDS} + S_{t,CDS}\left(\frac{1-e^{-\tau/\lambda_{t,CDS}}}{\tau/\lambda_{t,CDS}}\right) + \varepsilon_{\tau,CDS}, \tag{3}$$

where $\varepsilon_{\tau,CDS} \sim N(0, \sigma^2_{\tau,CDS})$, $t = 1, \dots, n$.

Given our expectations of the system we also incorporate the Hungarian inflation rate as well as the EUR/HUF exchange rate into our model. The latter is required to enable us to convert forecasted Euro cash-flows to a HUF basis. Furthermore, recent developments in the issuance and pricing policy of the debt management office inspires a growing need to measure the demand for retail instruments among residents. Therefore, we also added the net retail demand as a factor instead of the previously used Eurozone inflation. The decreasing share of inflation-linked EUR denominated debt is no longer a visible issue, therefore dropping the Eurozone inflation rate seems a rational choice.

We observe the macro factors present in our model with measurement error, thus

$$mac_{t,j} = \widehat{mac}_{t,j} + \varepsilon_{mac_j}, \tag{4}$$

where $\varepsilon_{mac_j} \sim N(0, \sigma^2_{j,mac})$, $j \in \{cpi_{hun}, retail\ demand, eur/huf\}$.

Given equations (1)-(4) we have a total of 11 factors.

Let $Y_t = \left[y_{t,H}(\tau), y_{t,E}(\tau), mac_{t,cpi_{hun}}, \Delta(mac_{t,\text{retail demand}}), \Delta\log(mac_{t,eur/huf}), \log\left(y_{t,CDS}(\tau)\right)\right]'$ be the vector of our observations[4] while $X_t = [L_{t,H}, S_{t,H}, C_{t,H}, L_{t,E}, S_{t,E}, C_{t,E}, mac_{t,j}, L_{t,CDS}, S_{t,CDS}]'$ is the vector of unobserved state variables i.e. the factors at time t ($t = 1, \dots, n$).

[4] Both $y_{t,H}$ and $y_{t,E}$ contain maturities for $\tau \in \{1M, 2M, 3M, 6M, 9M, 12M, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y, 12Y\}$, while $y_{t,CDS}$ contains maturities for $\tau \in \{6M, 12M, 2Y, 3Y, 4Y, 5Y, 7Y, 10Y\}$.

Let S_t denote an unobserved, discrete-valued 2-state Markov-switching variable ($S_t \in \{1,2\}$) reflecting our prior assumption that the economy switches between two distinct regimes. The first regime ('normal regime') represents economic stability and growth, while the second one ('crisis regime') is connected to a more turbulent crisis situation.

The transition probabilities of the 2-state Markov process are given by

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \tag{5}$$

where $p_{ij} = Pr(S_t = j | S_t = i)$ and $p_{i1} + p_{i2} = 1$ for all $i \in \{1,2\}$.

The state-space representation of our dynamic linear model at time t can be written as

$$Y_t = H_t X_t + \theta_t, \tag{6}$$

$$X_{t+1} = C_{S_t} + F_{S_t}(X_t - C_{S_t}) + \varepsilon_t, \tag{7}$$

$$\begin{bmatrix} \theta_t \\ \varepsilon_t \end{bmatrix} \sim N\left(0, \begin{pmatrix} R_{S_t} & 0 \\ 0 & Q_{S_t} \end{pmatrix}\right) \tag{8}$$

with $t = 1, \dots, n$. The measurement equation (6) describes the relation between the data and our factors, where Y_t, θ_t are $m \times 1$ vectors, X_t, ε_t are $f_n \times 1$ vectors, m determines the dimension of the measurement variables, while f_n accounts for the number of factors in the model[5]. H_t is an $m \times f_n$ matrix that links the observed variables to the unobserved factors at time t, mainly containing the factor-loadings of equations (1)-(3). H_t does not depend on the Markov variable S_t, it is only dependent on the time-decay parameters, $\lambda_{t,k}$ $k \in \{H, E, CDS\}$.

The transition equation (7) describes the dynamics of the state variables at time t, where F_{S_t} $(f_n \times f_n)$ is the transition matrix describing the evolution of the factors, while C_{S_t} represents the steady state factor values in the transition equation, which can be derived from the usual state-space representation of the transition equation given by

$$X_{t+1} = \mu_{S_t} + F_{S_t} X_t + \varepsilon_t, \tag{9}$$

with μ_{S_t} denoted as the drift component and $C_{S_t} = \mu_{S_t}\left(I_{fn} - F_{S_t}\right)^{-1}$. Considering that the factors are easy to interpret in econometrics, the steady state values hold valuable information. While an analyst does not necessarily have expectations for a given element of the drift parameter μ, they may have one for the Hungarian CPI. Both C_{S_t} and F_{S_t} are dependent on the Markov-switching variable S_t. R and Q are covariance matrices with appropriate dimensions.

The (i,j)-th element of the parameters dependent on S_t can be described by the equation

$$f^k_{i,j,S_t} = f^k_{i,j,1} S_{1t} + f^k_{i,j,2} S_{2t}, \tag{10}$$

where k denotes the chosen parameter ($k \in \{C, F, R, Q\}$), with $S_{lt} = 1$ if $S_t = l$, and $S_{lt} = 0$ otherwise for $l \in \{1,2\}$.

We have two additional restrictions based on our assumptions. First, R_{S_t} is diagonal, thus at every time t ($t = 1, \dots, n$) the estimation error in our yield curve points and the measurement errors in the macro factors are uncorrelated. Second, we assume that the foreign factors can influence the domestic ones but not vice versa.

[5] The index in f_n is in no connection with the count of the observations; n.

4.2 Parameter estimation

This section deals with the problem of estimating the parameters of the system described in Section 4.1. The entirety of our parameter space is denoted by $\Theta \in \{H_t, F_i, C_i, Q_i, R_i, P, (X_t, S_t)\}$, where $i \in \{1, 2\}$ and $t \in (1, \dots, n)$. To generate the sequence $(X_t, S_t)_{1 \le t \le n}$ we turn to the work of Kim (1994). To sample for the parameters F_i, C_i, Q_i and R_i ($i \in \{1, 2\}$) we use the distributions suggested by Villani (2009) and Sugita (2008). The sampling of the transition probabilities P is a draw from a beta distribution based on the generated state-realizations $(S_t)_{1 \le t \le n}$ and on the P_{prior} assumption. To estimate the parameter H we use the Random Walk Metropolis Hastings algorithm, as H is only dependent on the time-decay parameters $\lambda_{t,k}$, $k \in \{H, E, CDS\}$, all other parameters are estimated within the framework of a Gibbs sampler.

4.2.1 Kim filter

The Kim filter is an extension of the Kalman filter (1960) which gives a solution to the parameter estimation problem of general dynamic linear models with a Markov-switching parameter in state-space representation form.

Let's suppose that all parameters of the model defined in Section 4.1 excluding the sequence (X, S) are known. Let $\mathcal{F}_t$ be the information available at time t. As defined in the Kalman filter the goal is to forecast X_t based on information available at time $t-1$. Let

$$X_{t|t-1}^{(i,j)} = E(X_t | \mathcal{F}_{t-1}, S_t = j, S_{t-1} = i) \tag{11}$$

be the conditional expectation of X_t based on $\mathcal{F}_{t-1}$ and $S_t = j, S_{t-1} = i$, and

$$P_{t|t-1}^{(i,j)} = E\left(\left(X_t - X_{t|t-1}^{(i,j)}\right)\left(X_t - X_{t|t-1}^{(i,j)}\right)' \middle| \mathcal{F}_{t-1}, S_t = j, S_{t-1} = i\right) \tag{12}$$

the mean squared error of the forecast. Then, conditional on $S_t = j, S_{t-1} = i$ at time t, the following equations define the Kalman filter algorithm for the system defined in equations (6),(9) and (8). Equations (13)-(16) define the phase of prediction, (18) and (19) the phase of updating, while (17) defines the so-called Kalman gain:

$$X_{t|t-1}^{(i,j)} = \mu_j + F_j X_{t-1|t-1}^{(i)}, \tag{13}$$

$$P_{t|t-1}^{(i,j)} = F_j P_{t-1|t-1}^{(i)} F_j' + Q_j, \tag{14}$$

$$\eta_{t|t-1}^{(i,j)} = Y_t - H_t X_{t|t-1}^{(i,j)}, \tag{15}$$

$$f_{t|t-1}^{(i,j)} = H_t P_{t|t-1}^{(i,j)} H_t' + R_j, \tag{16}$$

$$K_t^{(i,j)} = P_{t|t-1}^{(i,j)} H_t' \left(f_{t|t-1}^{(i,j)}\right)^{-1}, \tag{17}$$

$$X_{t|t}^{(i,j)} = X_{t|t-1}^{(i,j)} + K_t^{(i,j)} \eta_{t|t-1}^{(i,j)}, \tag{18}$$

$$P_{t|t}^{(i,j)} = \left(I_{f_n} - K_t^{(i,j)} H_t\right) P_{t|t-1}^{(i,j)}. \tag{19}$$

$X_{t|t}^{(i)} = E(X_t | \mathcal{F}_t, S_t = i)$ is the conditional expectation of X_t based on $\mathcal{F}_t$ and $S_t = i$, while $P_{t|t}^{(i)} = E\left(\left(X_t - X_{t|t}^{(i)}\right)\left(X_t - X_{t|t}^{(i)}\right)' \middle| \mathcal{F}_t, S_t = i\right)$ is the mean squared error of X_t conditional on $\mathcal{F}_t$ and $S_t =$

i. $\eta_{t|t-1}^{(i,j)}$ is the conditional forecast error of Y_t based on $\mathcal{F}_{t-1}$ and $S_{t-1} = i$, $S_t = j$, while $f_{t|t-1}^{(i,j)}$ is the conditional covariance of $\eta_{t|t-1}^{(i,j)}$.

To handle the increase in the number of cases at each iteration caused by the different states, Kim suggests the following equations to collapse the terms $X_{t|t}^{(i,j)}$ and $P_{t|t}^{(i,j)}$ to $X_{t|t}^{(j)}$ and $P_{t|t}^{(j)}$ respectively

$$X_{t|t}^{(j)} = \frac{\sum_{i=1}^{M} \Pr(S_t = j, S_{t-1} = i|\mathcal{F}_t) X_{t|t}^{(i,j)}}{\Pr(S_t = j|\mathcal{F}_t)}, \tag{20}$$

$$P_{t|t}^{(j)} = \frac{\sum_{i=1}^{M} \Pr(S_t = j, S_{t-1} = i|\mathcal{F}_t)\left(P_{t|t}^{(i,j)} + \left(X_{t|t}^{(j)} - X_{t|t}^{(i,j)}\right)\left(X_{t|t}^{(j)} - X_{t|t}^{(i,j)}\right)'\right)}{\Pr(S_t = j|\mathcal{F}_t)}. \tag{21}$$

Using these approximations, the problem becomes feasible. To approximate the probability terms in equations (20) and (21) one can use the Hamilton filter suggested by Kim and Nelson (Kim and Nelson 1999).

The second step of the Kim filter is smoothing, which is a backward recursion for all $t = n, \dots, 1$ with the calculation of the expectation and mean squared error of X_t conditional on $\mathcal{F}_n$.

The final step of the Kim filter is backward sampling from the sequence $(X_t)_{1 \le t \le n}$ step-by-step from the conditional distribution

$$X_t|\mathcal{F}_t, X_{t+1} \sim N\left(X_{t|t,X_{t+1}}, P_{t|t,X_{t+1}}\right), \tag{22}$$

where $X_{n|n} = E(X_n|\mathcal{F}_n)$, $P_{n|n} = Cov(X_n|\mathcal{F}_n)$ while $X_{t|t,X_{t+1}} = E(X_t|\mathcal{F}_t, X_{t+1})$ and $P_{t|t,X_{t+1}} = Cov(X_t|\mathcal{F}_t, X_{t+1})$. Assuming that X is stationary, the unconditional mean and covariance matrix and the steady state probabilities can be used to start the filter. We will not go into any more detail, but a step-by-step explanation of the Kim filter can be found in Kim and Nelson (1999).

4.2.2 Parameter sampling

With the assumption that H, R, C, F and Q parameters are known for both regimes, the Kim filter is applicable to sample the values of X_t, and Kim and Nelson (1999) also suggest a way to sample S_t for all $t = 1, \dots, n$.

To get a sample for the parameters R, C, F and Q for a given $(X_t, S_t)_{1 \le t \le n}$ we use the distributions suggested by Sugita (2008) and Villani (2009). Our model is equivalent to the one used by Sugita. We can easily see this for example by substituting $k, q = 1$, $p = f_n$, $\Pi = F$, $d_t \equiv 1$, $\Phi = \mu$, $\psi = C$ and $\Sigma = Q$ into the notations of equation (7).

The selection of the starting covariance matrices V_0 and Σ_0 is crucial in a sense that it determines the parameter space explored.

Conditional on $(S_t)_{1 \le t \le n}$ P is independent of the dataset and the model's other parameter. Assuming independent beta distributions for the priors $p_{11} \sim beta(l_{11}, l_{12})$ and $p_{22} \sim beta(l_{22}, l_{21})$ Kim has shown that

$$p_{11}|\text{S} \sim beta(l_{11} + k_{11}, l_{12} + k_{12}), \tag{23}$$

$$p_{22}|\text{S} \sim beta(l_{22} + k_{22}, l_{21} + k_{21}), \tag{24}$$

where k_{ij} refers to the transitions from state i to j, which can be easily counted given $(S_t)_{1 \le t \le n}$.

To estimate the parameter H we use the Random Walk Metropolis Hastings algorithm, as H is only dependent on the time-decay parameters $\lambda_{t,k}$, where $k \in \{H, E, CDS\}$. $\lambda_{t,k}$ is determined using the $\lambda_{t,k}^{new} = \lambda_{t,k} + \kappa_t \varepsilon_{t,k}$ equation, where $k \in \{H, E, CDS\}$, κ_t can be arbitrarily chosen and $\varepsilon_{t,k} \sim N(0,1)$ for every t and k. Using the generated parameter we calculate a candidate H, which we keep with a calculated α probability, after the evaluation of the measurement equation's likelihood function.

4.2.3 Initial values

The first step in defining the initial set of values is the setting of X^0, which entails the substitution of the empirical values. For example, the level parameter (which is the short term level in our case) is the shortest available point of the yield curve. The observed data defines the matrix Y. H^0 is defined by the observed maturities, with the assumptions that $\lambda_{0,k} = 1$ and $k \in \{H, E, CDS\}$. R^0 can be calculated from X^0, Y and H^0.

Next step is the estimation of the parameters of the transition equation with the ordinary least square (OLS) method. A simple Kalman filter is used to improve the estimates. It is crucial to mention that so far no regime switching is included.

The prior transition matrix P_{prior} is based on expert estimates. Based on the number of observed periods (n) it defines the share of each regime on the observed time horizon. Squared errors for each period can be calculated based on the initial values above substituted in equations (6) and (7). Sorting the periods based on the squared errors define S^0. The empirical covariance matrices of the separated periods define the initial covariance matrices R_i^0 and Q_i^0, $i \in \{1, 2\}$. $F_i^0 = diag(F^0)$, $i \in \{1, 2\}$ is the modified Minnesota prior, while $C_i^0 = \bar{X}^0_{t \in \{S^0 = i\}}$ ($i \in \{1, 2\}$) or can be modified by expectations of the analyst.

4.2.4 The algorithm

Our parameter estimation algorithm is based on the works of Blake & Mumtaz (2012). The algorithm consists of the following steps:

1. Initialization. Setting the initial values of the sequence $(X_t, S_t)_{1 \le t \le n}$ and the parameters F_i^0, C_i^0, Q_i^0, R_i^0, $i \in \{1, 2\}$.
2. Sampling from $(X_t, S_t)_{1 \le t \le n}$. $(X_t, S_t)_{1 \le t \le n}$ is generated based on the steps of the Kim filter conditional on F_i, C_i, Q_i, R_i, $i \in \{1, 2\}$ and P.
3. Sampling P. P is drawn from a beta distribution based on the generated state-realizations $(S_t)_{1 \le t \le n}$ and P_{prior}.
4. Sampling F_i, C_i, Q_i and R_i, $i \in \{1, 2\}$ based on the appropriate distribution defined by Villani (2009).
5. Sampling H_t, $\lambda_{t,k}$, $t \in \{1, \dots, n\}$ and $k \in \{H, E, CDS\}$ using Random Walk Metropolis Hastings algorithm.
6. Repeat steps 2-5 until the convergence meets our expectations.

5. Calculation of costs and risks

In this section we address the issues regarding the calculation of different cost and risk factors. Quantifying costs and risks is important because our goal of creating Pareto-optimal composition of borrowing requires us to set objective, measurable criteria on which different composition of borrowing can be compared and ranked.

5.1 Cost factors

Cost factors can be divided into 4 groups. The most prominent cost factor is the interest expenditure, which is the cumulated sum of the coupon payments of all instruments. The second cost factor is the issuance price discount or premium (loss or gain). Treasury Bills do not pay coupons whereas all other instruments are issued exactly at, or closely to, *par* in the model. The third cost factor is the currency cost, which boils down from the deviation of the EUR/HUF exchange rate. In accordance with the strategic goal of the Hungarian DMO, all foreign liabilities are either issued in, or swapped into, Euro. The model calculates all costs in HUF (based on Hungarian accounting rules), therefore currency cost is a key factor in determining the optimal share of FX debt in the portfolio. The fourth cost factor is the so-called liquidity premium, which arises when excessive issuance of government bonds or treasury bills is needed to fulfill the funding requirements. We address this issue in Section 5.1.1.

All costs are aggregated in HUF over the time horizon on every trajectory using accrual accounting. Such time-proportional accounting applies for all coupon payment dates in the timeframe, for all instruments. Smoothing the costs on the time horizon makes the borrowing compositions (and different time horizons) comparable to each other. It also mitigates the importance of whether a payment date is before or after the model's timeframe.

If all the simulated trajectories are taken into account, we get the stochastic distribution of the costs. We call the expected cost of the stochastic distribution of the cumulated costs the cost of the specific borrowing composition. Minimizing expected cost is one of the objective functions we use for optimization.

5.1.1 Funding liquidity considerations

It is clear that meeting the funding needs by issuing in only one or two instruments is impossible due to limits of market demand, and it is also clear that excess issuance comes with a cost. Avoiding borrowing compositions that encourage excess issuance is desired by the DMO. The introduction of so-called liquidity constraints and liquidity premium arises as a possible solution to handle the cases of excess issuance.

We turn to the Hungarian DMO auction database to calculate issuance limits at a given time with the use of market demand. We calculate two values for every wholesale instrument. One representing the maximum amount to be issued at a given time at normal 'market' price, which we call 'liquidity threshold', while 'liquidity constraints' are introduced as the maximum available amount of an instrument to be issued at a given time. Therefore, liquidity constraints play an important role in the issuance algorithm of the simulation framework, detailed in Section 6.3. 'Liquidity premium' is introduced as the cost premium for the DMO to be paid for the amount issued above the liquidity threshold.

We assume that Discount T-Bills do not have a liquidity constraint. In practice, investors tend to buy short-term instruments even in periods of high volatility, although with a significant liquidity premium. Therefore, Discount T-Bills can be viewed as safety instruments in times of distress when funding requirements cannot be met with the issuance of other instruments. We introduce an indicator similar to the liquidity risk premium of Drehmann and Nikolaou (2013). Our indicator measures the difference between the rejected bids and the marginal rate of the analyzed auctions. Let $E(h_t)$ be the expected marginal rate of the auction (i.e. the highest accepted yield) held at time t. Let

$$\iota_{b,i,t} = \left(\rho_{b,i,t} - E(h_t)\right) v_{b,i,t} \quad (25)$$

be the *adjusted rejected bid* on the auction held at time t, by the auction participant $i \in \{1,2,\dots,N\}$. The bid $b \in \{1,2,\dots,B\}$ stands for the amount $v_{b,i,t}$ and rate $\rho_{b,i,t}$, and $E(h_t) < \rho_{b,i,t}$. Which means that $\iota_{b,i,t}$ is the premium expected by the bidder for the buying of an additional $v_{b,i,t}$ amount above the marginal rate. The liquidity premium at time t is the aggregate of the adjusted rejected bids

$$lp_t = 100\frac{\sum_{i=1}^{N}\sum_{b=1}^{B}\iota_{b,i,t}}{E(\bar{v}_t)} \tag{26}$$

in basis points, where $\bar{v}_t$ denotes the cumulated amount of unsuccessful bids. Liquidity premium thus gives us a quantified value of the investor's willingness of providing additional liquidity.

The liquidity premium can be quantified for all instruments and regimes, and then used to measure excess issuances. The liquidity premium formula above measures the increase in yield when the total issuance from one instrument in a month equals the liquidity constraint. The cost increase for that particular instrument also spills over to following months, simulating rising yields in case of excess issuance. Naturally, the liquidity premium can be increased further if there is excess issuance in the following month as well. This accrued liquidity premium decays exponentially on a monthly basis. We usually use a decay factor of 0.9.

5.2 Risk factors

Risks of a government bond portfolio can be measured in multiple ways. We separated risk metrics into three main categories. Volatility metrics, such as standard deviation, are adequate benchmarks for the risks arising from interest rate or exchange rate changes throughout the simulation horizon. However, they are, by definition, backward-looking measures. A portfolio having a low standard deviation throughout the simulation time horizon does not necessarily mean it is safe from future interest rate changes. The second category, interest rate sensitivity metrics, addresses this issue. Measures like Average Time to Re-fixing (ATR) tell us something about the inherent interest rate sensitivity of a portfolio, and can be viewed as a forward-looking measure. The third category, refinancing risk also contains forward-looking measures. A high refinancing ratio tells us that there could be serious funding liquidity issues in the future. Compressing the complex risk characteristics of a debt portfolio into a single number *ex-ante* would only be possible by assigning weights to different risk metrics. However, our intention was to allow the decision maker to do this *ex-post*. Choosing a single risk metric gives us too little information about the risk characteristics of a portfolio, while attempting to use every relevant measure makes the optimization space too large. Our solution for this dilemma was using one risk measure from each of the three categories as objective functions in our optimization scheme.

5.2.1 Volatility metrics

To measure the risks rising from the volatility of the yield curve, we calculate the standard deviation, the Cost-at-Risk and the Conditional Cost-at-Risk at different confidence levels of the stochastic distribution of costs (basically an implementation of the Monte Carlo simulation for calculating VaR or CVaR). The corrected sample standard deviation is well known. Let $c_{1,s} \leq c_{2,s} \leq \cdots \leq c_{r,s}$ denote the ordered costs of r trajectories based on s borrowing composition, then we calculate the Cost-at-Risk at $0 < \alpha < 100$ confidence level as

$$CaR_\alpha(s) = c_{\left\lfloor r\frac{\alpha}{100}\right\rfloor}, \tag{27}$$

while the Conditional Cost-at-Risk at $0 < \alpha < 100$ confidence level is

$$CCaR_{\alpha}(s) = \frac{1}{\left\lfloor r\left(1-\frac{\alpha}{100}\right)\right\rfloor+1}\sum_{i=\left\lfloor r\frac{\alpha}{100}\right\rfloor}^{r} c_i. \tag{28}$$

The Cost-at-Risk at confidence level α tells us that based on the worst $(100-\alpha)\%$ of scenarios how much the minimum cost will be in the chosen timeframe, while the Conditional Cost-at-Risk at confidence level α tells us about the expected costs of the worst $(100-\alpha)\%$ of scenarios in the chosen timeframe.

Although measuring the deviation of costs is crucial, there are several other metrics to assess risks. We address this issue in the following sections.

5.2.2 Interest rate sensitivity

Changes in the yield curve may cause increase in the interest rate expenditures, and therefore it is crucial to measure interest rate sensitivity from a debt management perspective. Short-term or variable interest rate instruments are considered riskier than long-term or fixed rate instruments in this aspect. Duration and ATR can be used to capture the exposure of the debt portfolio to changes in interest rates.

The ATR of a fixed interest rate instrument is equal to its time to maturity, while for a floating rate instrument it measures the time until its next coupon re-fixing. Thus the ATR of the debt portfolio measures how long on average it takes for the interest rate changes to be incorporated into the costs of the overall debt. Let

$$atr_t = \frac{\sum_{i=1}^{N}\zeta_{t,i}v_{t,i}}{\sum_{i=1}^{N}v_{t,i}} \tag{29}$$

denote the ATR of the debt portfolio at time t, with N instruments present in the portfolio. $\zeta_{t,i}$ denotes the time to the next coupon fixing at time t for instrument i, while $v_{t,i} = FV_iP_{t,i}\left(\frac{100-\sum_{k\leq t}C_{k,i}}{100}\right)$, where FV_i is the Face Value of instrument i, $P_{t,i}$ denotes the pieces of instruments i issued by the DMO at time t, and $C_{k,i}$ is the capital repaid at time k of instrument i given in percentage points. A higher value of ATR indicates a higher immunity against interest rate changes, while low values indicate that a high share of the debt portfolio will be subject to interest rate changes in the near future, thus signaling a higher exposure to interest rate risk.

5.2.3 Refinancing risk

The risk rising from the maturity structure of a given portfolio is called refinancing risk. Refinancing risk measures the share of the debt portfolio maturing in a given time period. Generally, in periods of turbulence or high volatility, market liquidity typically dries up, making the renewal of maturing debt expensive, or, in an extreme case, impossible. Therefore, it is vital for debt management to smooth the maturity structure, and thus avoid maturity peaks. Average Term-to-Maturity (ATM) measures how long it takes on average to renew outstanding debt. Let

$$atm_t = \frac{\sum_{i=1}^{N}\eta_{t,i}v_{t,i}}{\sum_{i=1}^{N}v_{t,i}} \tag{30}$$

denote the ATM of the outstanding debt portfolio at time t, with N instruments present in the portfolio. $\eta_{t,i}$ denotes the remaining term to maturity at time t for instrument i, while $v_{t,i}$ is the same as above. Refinancing risk is lower when ATM is higher, as it means a lower renewal rate in the short term.

The refinancing ratio measures the share of the debt portfolio to be renewed on a given time horizon. Let

$$refin_{t,\lambda} = \frac{\sum_{i=1}^{N} v_{t,i} \mathbf{1}_{\{\eta_{t,i} \leq \lambda\}} l_i}{\sum_{i=1}^{N} v_{t,i}} \quad (31)$$

be the refinancing ratio of the outstanding debt portfolio at time t, with time horizon λ. $\mathbf{1}_{\{\eta_{t,i} \leq \lambda\}}$ denotes the indicator function of whether instrument i at time t matures within the chosen time period, and $l_i = 1/2$ if i is retail, while 1 in every other case. We assume – and years of experience at the Hungarian Government Debt Management Agency supports – that retail investors' willingness to renew their government securities holdings tend to be higher than that of wholesale investors, especially in times of high volatility. As Hungarian retail rates are significantly higher than wholesale in a low interest rate environment, holding the retail rates constant is viable even as wholesale rates rise. Experience suggests that in a high volatility, high interest rate environment, retail investors tend to buy retail instruments even if wholesale alternatives are strictly superior. This stems from the strength of the retail brand and the general lack of financial knowledge among retail investors. Even though retail instruments can be redeemed under favorable conditions (98% to par depending on the instrument), the prevalence of inflation-linked and step-up coupon bonds may deter retail investors from doing so.

Keeping the refinancing ratio at a low level stabilizes the debt service, increases trust in investors and supports higher ratings given by credit rating agencies.

6. Simulation Framework

In this section we give an overview of the simulation and optimization process, including the selection of planned instruments, the issuance algorithm and the optimization algorithm. We also analyze the optimal allocation of resources (number of borrowing compositions and iterations) for the optimization algorithm.

6.1 Planned instruments

To analyze the characteristics of the costs and corresponding risk factors of the Hungarian debt portfolio we simulate issuances of chosen instruments on the specified time horizon. We categorize all portfolio instruments into three separate groups, namely the domestic wholesale, the FX, and the retail debt. There are several model instruments to choose from in each market segment:

Wholesale segment:

- 3-, 5- , 10- and 15-year fixed interest rate Government Bonds
- 5-year floating interest rate Government Bond
- 12-month Discount Treasury Bill

FX segment:

- 5- and 10-year fixed and floating interest rate, Euro denominated Bonds

Retail segment:

- 1-year Hungarian Government Security ("1MÁP")
- Premium Hungarian Government Securities (3- and 5-year inflation-linked bonds, "PMÁP")

- Hungarian Government Bond Plus (5-year step-up coupon bond, "MÁP+")

All model instruments share the same characteristics and properties as those issued by the Hungarian DMO. The range of model instruments can be easily changed by adding or removing instrument types.

The issuance of retail securities is an external factor concerning the optimization framework. Retail issuances are trajectory dependent as described in Section 4.1, but are independent from the borrowing composition. A higher debt-renewal willingness is also assumed for the retail segment based on the DMO's experience (as described in Section 5.2.3).

In the model, Government Bonds are issued three times a year, with the opportunity of a monthly reopening, while Treasury Bills are issued each month. FX bonds are issued yearly with a trajectory dependent choice for the issuance month[6], and are not reopened. Premium Hungarian Government Securities are issued at the start of every calendar year, with the opportunity of a monthly reopening. The other two retail securities are issued every month in the model.

The initial portfolio to start the simulation is identical to the actual government debt portfolio, with the exception that all actual instruments are listed as one of the above model instruments. Thus, the model is able to generate accurate and realistic results that can be used directly to set benchmarks for the government debt.

6.2 Optimization

The purpose of optimization is to construct composition of borrowing that minimize the costs and risks of the debt portfolio. We chose three different risk metrics: standard deviation of the costs, 1-year refinancing ratio and ATR. We intend to maximize the latter and minimize the former two as well as the expected costs. These goals are contradictory, therefore finding the set of Pareto-optimal solutions is our objective.

To achieve this, we use a multi-objective optimization algorithm applying evolutionary heuristics to the problem. We implement the algorithm of Sun, Zhu and Cai (2017). The algorithm finds the corner solutions first and fills the optimization space afterwards. The output of the optimization algorithm is a set of Pareto-optimal solutions. A two-dimensional Pareto-front can then be established using posterior weighting of the risk metrics, expressing the preferences of the decision maker. Using this method, we can produce a traditional mean-variance efficient frontier or plot the expected costs versus an arbitrarily weighted composite risk measure.

Similar to traditional computational optimization, our implementation relies on multiple iterations (repeated evaluations of the objective function) to approach the optimum. However, to attain multiple Pareto-optimal solutions, we continuously work with a set of inputs, consisting of a large amount of borrowing compositions. Borrowing compositions constitute the input of the issuance algorithm.

The objective function G can be mathematically expressed as:

$$\min_{f.h\ \sum_s=1;\forall s_i\in s:\ 0\le s_i\le 1} G(s) = (cost(s), std(s), refin(s), -atr(s)) \quad (32)$$

[6] In practice, the Government Debt Management Agency has a degree of freedom in deciding when to issue FX debt. This is modeled by a single issuance per calendar year. In the model, FX debt is issued in the first month where the level of Euro area yield curve declines for two months in a row, or in December if the criterion is not met throughout a given calendar year.

where s is a borrowing composition, serving as an input for the issuance algorithm. In the first iteration of the optimization process, each borrowing composition is randomized, taking a uniform distribution on the simplex of the appropriate dimension. As detailed in Section 6.1, we optimize for 10 (6 domestic wholesale and 4 FX) planned instruments, giving us 9 degrees of freedom.

6.3 Issuance algorithm

The issuance algorithm is an essential part of the optimization process, connecting input (borrowing composition) and the output (costs and risks). Issuances are simulated with a monthly frequency. The horizon of the simulation is the first calendar year after the start plus 5 years, adding up to a total of 60 to 71 months.

An important element of the simulation of issuances is the handling of the Treasury Single Account (TSA). The TSA is an account belonging to the Hungarian State Treasury, held at the Central Bank. In case of new issuances, the inflows are deposited on the TSA, while debt charges are paid from the TSA (beside other budgetary revenues and expenditures). The Hungarian Government Debt Management Agency actively manages the TSA to have a liquidity buffer in case of larger debt redemptions. In the model, new issuances increase the TSA balance, while redemptions and the budget deficit decrease it. In the model, interest payments do not decrease the TSA balance, because we assume that the externally given budget deficit includes interest expenditures as well. We call an n-month TSA requirement the total outstanding amount of government securities maturing within the next n months. Maturing Discount Treasury Bills, having a quasi-infinite demand in the model, are excluded from the TSA requirement, while retail instruments – assumed to have an inherently lower refinancing risk – have a weight 0.5 in the TSA requirement calculation. We designed the issuance algorithm so that the TSA account balance should fall in a given range. We set the minimum TSA balance as the 2-month TSA requirement and the maximum as the 4-month requirement.

Now let d be the number of simulated months, and $\mathrm{v} = [v_0\ v_1\ v_2\ \dots\ v_d]'$ the total outstanding amount of debt instruments minus the initial TSA balance we want to attain. Additionally, let v_0 be the outstanding amount of debt minus the TSA balance at the start of the simulation, both attained from factual data. Using the vector of yearly projected deficits, we create a monthly $\mathrm{f} = [f_1\ f_2 \dots f_d]'$ vector. We can then recursively calculate $v_i = v_{i-1} f_i$, $(i = 1, 2, \dots, d)$. We also create a v^{max} maximal outstanding amount vector, where $\mathrm{v}_{\mathrm{i}}^{max} = v_i + TSA_i^{max}, (i = 1,2, \dots, d)$, and a v^{min} minimal outstanding amount vector, where $\mathrm{v}_{\mathrm{i}}^{min} = v_i + TSA_i^{min}, (i = 1,2, \dots, d)$. Naturally, the TSA^{min} and TSA^{max} vectors contain the minimal and maximal TSA requirement for every t. It has to be noted, that TSA requirements, and therefore the minimal and maximal debt amount vectors, depend on the given borrowing composition and trajectory.

As in reality, large maturing amounts can exist in the model, making it difficult to refinance them due to liquidity constraints. The usual solution in reality, used in the model as well, is pre-financing. In case a security is about to mature in the near future, we increase the amounts to issue from that type of instrument to avoid refinancing it in a single month. We do not use pre-financing for Discount Treasury Bills issued monthly. Most other domestic instruments are pre-financed 6 months in advance, while FX issuances can be pre-financed starting January the calendar year before maturity in order to have at least one FX issue date before maturity for certain.

If we intend to reach the desired composition of debt in m months (we generally use $m = 12$), let $\mathrm{w} = [w_1\ w_2 \dots w_d]'$ be a vector of weights calculated by:

$$w_i = \begin{cases} i/m, & i < m \\ 1, & i \geq m \end{cases}. \tag{33}$$

Furthermore, let p denote the shares of the initial portfolio and T a $\mathrm{d} \times \mathrm{n}$ matrix, where

$$T_{ij} = v_i^{min}\left((1 - w_i)p_j + w_i s_j\right). \tag{34}$$

The value of T_{ij} gives the desired outstanding amount of instrument j for month i.

To calculate the net issuance of instruments for month i, let t denote the i-th row of T, u the current outstanding amounts of instruments at month i and u' the vector of amounts to be pre-financed. The net issuance vector x is the result of

$$\begin{aligned}
&\underset{x}{lexmax}\left(min\left(\frac{x + u - u'}{t}\right)\right)\\
&s.t.: \quad x \geq 0,\\
&x \leq max(0\,, t - u + u'),\\
&Lx \leq b,\\
&1_n'(x + u) \leq max(v_i^{max}, 1_n' u),
\end{aligned} \tag{35}$$

where b is the vector of liquidity constraints and L is the corresponding summary matrix. We use 5 different liquidity constraints: FX, Discount Treasury Bill and 3/5/10/15-year Hungarian Government Bond. Retail issuances are treated as external and calculated separately, before the issuance algorithm.

The so-called lexicographic maximization, inspired by the Constrained Equal Losses rule (Fleiner and Sziklai 2012), can be interpreted as trying to maximize the smallest share of the actual versus the desired outstanding amount. If it can be accomplished in multiple ways, we maximize the second smallest share and so on.

If the solution x does not fulfill an additional $1_\mathrm{n}'(\mathrm{x} + \mathrm{u}) \geq v_i^{min}$ criterion, meaning that the minimal total outstanding amount is not reached, we finance the remaining value by issuing Treasury Bills, even if its liquidity constraint is exceeded. This simulates emergency borrowing at a significantly higher cost.

6.4 Meta-optimization

Meta-optimization is the process of optimizing the optimization algorithm. In our case, this is essentially a resource allocation problem. We have a finite temporal constraint, wanting to limit the optimization process to 2-3 hours so that results can be calculated and analyzed in one day. This means that the number of borrowing compositions times the number of iterations is limited to about 5 million. We also have a "soft" limit on the number of borrowing compositions (approximately $50{,}000$), as the optimization algorithm is quadratic in memory usage. We use several metrics to measure the performance of different borrowing composition and iteration distributions.

Table 1: Resource allocation descriptive statistics

No. FinComp/Iter	**2,500/ 2,000**	**5,000/ 1,000**	**10,000/ 500**	**25,000/ 200**	**50,000/ 100**
Minimum cost (%)	3.9955	3.9948	3.9949	3.9949	3.9949
Minimum STD (%)	0.3054	0.3054	0.3054	0.3054	0.3054
Maximum ATR (years)	4.8584	4.8584	4.8584	4.8583	4.8584

Minimum share of 1-year refinancing (%)	10.3012	10.3140	10.3313	10.2972	10.3174
Average cost (%)	4.0917	4.0884	4.0885	4.0890	4.0964
Average STD (%)	0.5101	0.5076	0.5056	0.5038	0.5088
Average ATR (years)	3.9990	3.9606	3.9714	3.9766	4.0063
Average share of 1-year refinancing (%)	12.3035	12.2596	12.2364	12.1900	12.0550
Hypervolume	557.63	558.39	558.66	559.92	560.16

Source: Own calculations

Table 1 shows a comparison of different resource allocation options. We examine numbers of borrowing compositions ranging from 2,500 to 50,000 and the corresponding iteration number, keeping a constant budget. The corner solutions are quite similar for every option, with the 25,000 borrowing composition one having a slight edge in finding the borrowing composition with the minimum share of 1-year refinancing. There are slight variations in the average objective function values, but it shows no conclusive evidence for the superiority of any allocation.

The hypervolume metric deserves some more explanation. Also known as the Lebesgue measure, this is the n-dimensional equivalent of the 3-dimensional volume. This is one of the most widely used and accepted measure when comparing different multi-objective optimization algorithms (Riquelme, Von Lücken and Baran 2015). We use the hypervolume to calculate the size of the covered optimization space with regards to a reference point. The hypervolume calculations were made using the algorithm of Fonseca, Paquete és López-Ibánez (2006). We use the so-called nadir point as the reference point, which can be described with a vector of the worst possible objective function values, and is shown in Table2.

Table 2: The nadir point

Maximum cost (%)	Maximum STD (%)	Minimum ATR (years)	Maximum share of 1-year refinancing (%)
8.41	2.46	1.86	30.73

Source: Own calculations

The nadir point was calculated by optimizing for maximum cost, maximum standard deviation, minimum ATR and maximum share of 1-year refinancing. The high costs were the result of issuing almost 100% Treasury Bills, the short end of the yield curve rising as a consequence, accruing excess liquidity cost.

The hypervolume comparison in Table 1 shows that increasing the number of borrowing compositions leads to greater coverage despite having fewer iterations to refine those compositions. We can also make pairwise comparisons between the different variants.

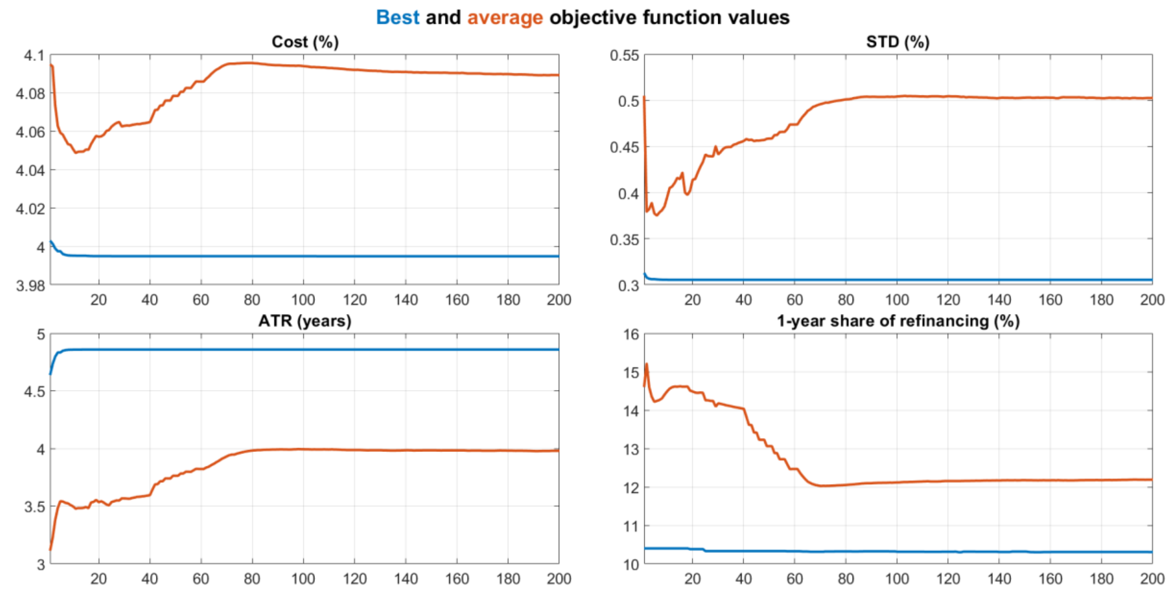
Table 3: Pairwise comparisons of Pareto-optimality

No. FinComp/Iter	2,500/2,000	5,000/1,000	10,000/500	25,000/200	50,000/100
2,500/2,000	100.00	79.00	63.88	46.72	56.20
5,000/1,000	97.08	100.00	80.08	62.66	67.76
10,000/500	99.01	96.43	100.00	77.20	77.81
25,000/200	99.66	98.66	95.66	99.97	89.39
50,000/100	96.33	93.00	88.39	80.56	99.93

Source: Own calculations, percentage points

Table 3 shows pairwise comparisons based on Pareto-optimality. The main diagonal shows the percentage of Pareto-optimal borrowing compositions of different borrowing compositions and iteration variants. This is very close to 100% in every case, which means that by the end of any optimization, almost all borrowing compositions were Pareto-optimal with regards to that specific pool of borrowing compositions. The non-diagonal elements of Table 3 show pairs of optimization results pooled together. The numbers tell the percentage of borrowing compositions from the row element that are Pareto-optimal in the combined pool. For example, if we pool together the $5{,}000$ borrowing composition optimization with the $25{,}000$ one, we get a total of $30{,}000$ borrowing compositions. If we measure Pareto-optimality in the combined pool of $30{,}000$, a total of 98.66% of the $25{,}000$ borrowing compositions are Pareto optimal (despite being optimized through only 200 iterations), while only 79% of $5{,}000$ borrowing compositions are optimal (despite being optimized through $1{,}000$ iterations). Table 3 shows that increasing the number of borrowing compositions is a worthwhile trade-off to a certain degree. However, comparing $25{,}000$ and $50{,}000$ borrowing compositions, the former seems to be better based on this data. Therefore, we can say that 100 iterations are not enough for the optimization algorithm to converge.

Figure 1: Evolution of objective function values



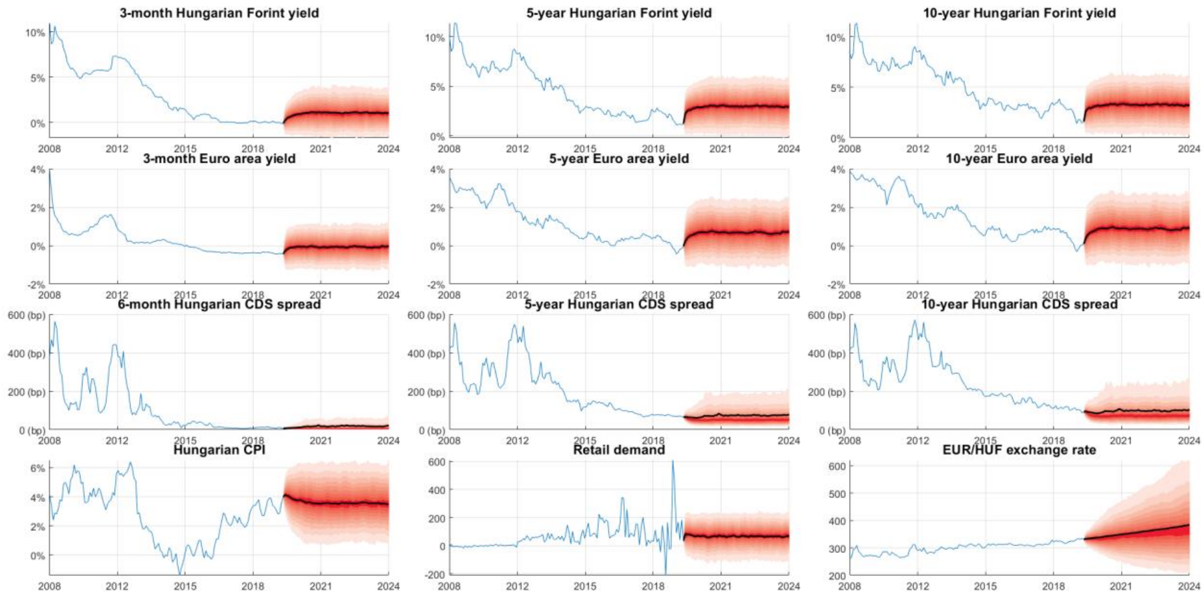
Source: Own calculations

Figure 1 shows the evolution of the best and average objective function values of $25{,}000$ borrowing compositions through 200 iterations. One can see that the best objective function values converge in only a handful of iterations. This is unsurprising as the used optimization algorithm (Sun, Zhu and Cai 2017) finds the corner solutions first. However, there are large changes in the average objective function values after the algorithm finds the corner solutions and starts to diversify the Pareto-optimal compositions, covering more and more of the optimization space. One can see that changes continue on even after 100 iterations, with borrowing compositions getting more refined. This can explain why $25{,}000$ borrowing compositions with 200 iterations was the standout performer in the pairwise Pareto-optimality comparison.

7. Results

In this section we present and analyze the results of forecasting and optimization.

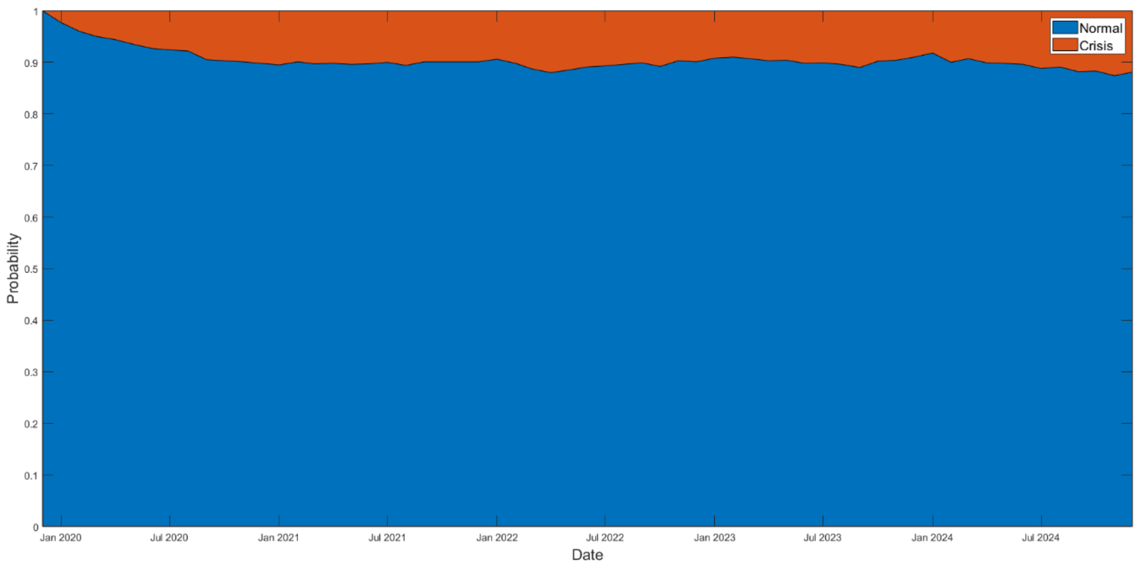
Figure 2: Evolution of yields and macro factors



Source: Own calculations

Figure 2 shows selected forecasts, including yields and macro factors, based on the results of the Markov regime switching model. Short Hungarian Forint yields are expected to rise to the 1% level, but remain negative in several trajectories. Long Hungarian Forint yields are expected to reach the 3.5% level. We expect Euro area yields to rise as well according to our forecast. Our medium-term expectation for the Hungarian CPI is 3.5%, while Hungarian CDS spreads are expected to remain stagnant for the next 5 years. Perhaps the most impactful result is the expectation that the EUR/HUF exchange rate will exceed 380 on average by the end of 2024. This means approximately a 3% yearly devaluation of the Hungarian Forint versus the Euro, raising the cost of issuing FX debt. Net retail demand is expected to stay positive throughout the simulation horizon.

Figure 3: Forecasted state probabilities



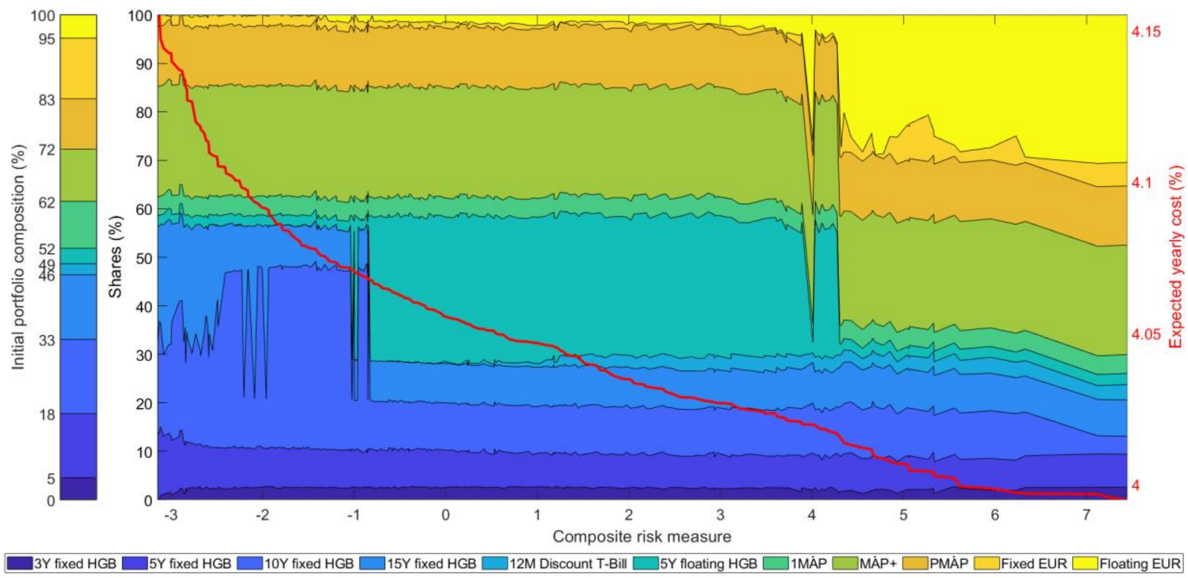
Source: Own calculations

Figure 3 shows the forecasted state probabilities. We are currently in a normal regime, but as we move forward in time, there is always a chance of entering the crisis regime. There is, according to our forecast, approximately a 10% chance of being in the crisis regime for most of the simulation horizon.

We used $25{,}000$ borrowing compositions for optimization and refined them through 200 iterations. We aggregated costs and risks from December 2019 to December 2024. Two-dimensional cost-risk frontiers can be constructed either by selecting a singular risk metric, using an arbitrary weighting scheme for the different risk measures. Our method of calculating a “composite risk measure” given the weights for different risk measures is standardizing every risk metric within the pool of $25{,}000$ borrowing compositions, and then taking the sum with appropriate weights in mind. The ideal way to use the model is to have the decision maker set the actual weights.

In this paper we show the results of setting equal weights for all three risk measures. Using this particular weighting scheme, 234 borrowing compositions are Pareto-optimal in a cost-composite risk trade-off.

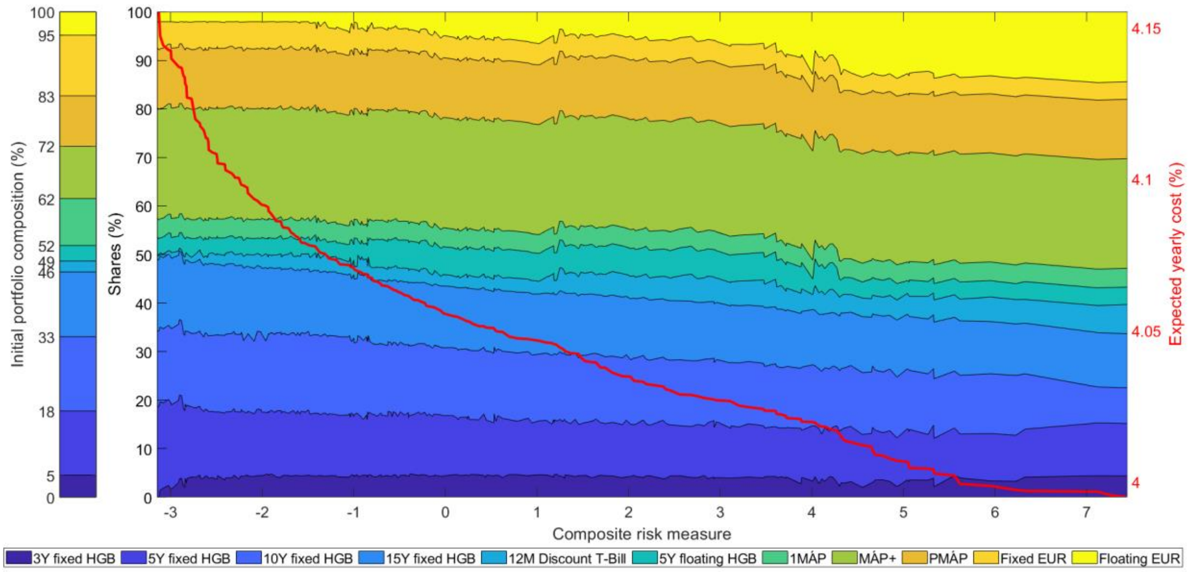
Figure 4: Optimal borrowing compositions, equal weighted standard deviation, ATR, 1-year share of refinancing



Source: Own calculations

Figure 4 shows the optimal borrowing compositions and the expected yearly cost of these compositions. We can also see the initial (December 2019) portfolio composition sorted into the most appropriate model instruments. The expected yearly costs of the optimal borrowing compositions have a tight range of $4 - 4.15\%$. The share of retail debt is mostly independent from the borrowing composition, treated as an external feature the debt manager has no control over. The expansion of retail funding is a cornerstone of Hungary’s current economic policy and therefore controlling the retail premium to curb demand is not an option for the debt manager. The share of retail debt goes from around 30% to almost 40% of the total debt during the simulation horizon. FX debt is not preferred by the model, almost only the high-risk borrowing compositions have measurable FX issuance. The borrowing compositions seem “choppy” on occasion, with the 5-year floating government bond taking up a large share in mid-risk compositions and almost vanishing in low-risk or high-risk ones. This behavior merits further explanation.

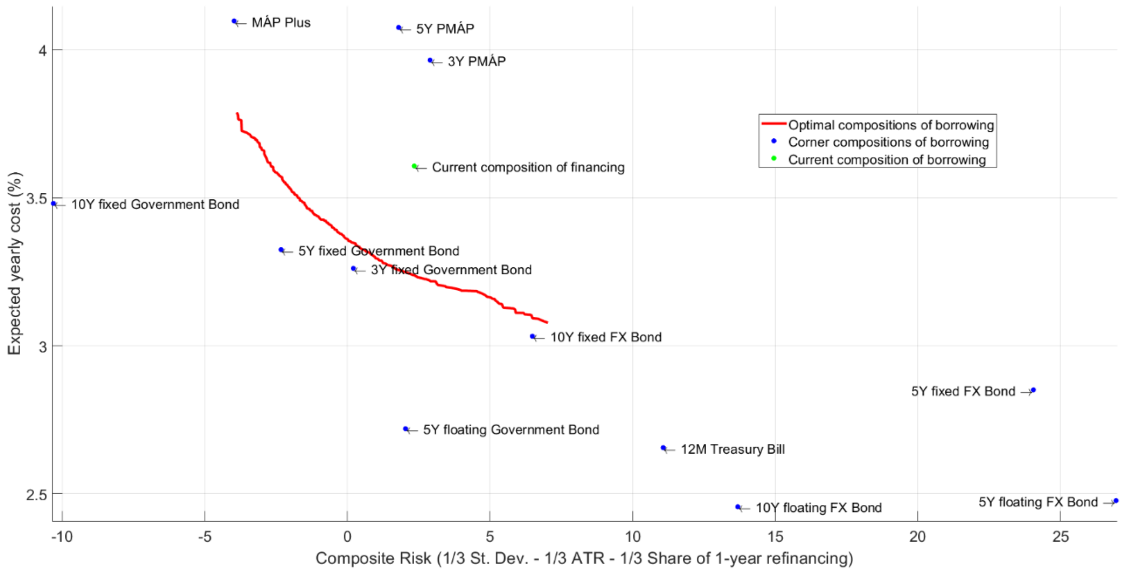
Figure 5: December 2024 portfolios, equal weighted standard deviation, ATR, 1-year share of refinancing



Source: Own calculations

Figure 5 shows the actual portfolios to be reached by December 2024. We can see that the transitions between different instruments are much more gradual compared to borrowing compositions. The real difference between low-risk and high-risk portfolios is a gradual decrease of long-maturity fixed interest rate bonds and an increase in T-Bills and floating rate FX bonds. We can also see that despite the existence of borrowing compositions with a large domestic floating rate bond share, it never reaches more than about 5% of the portfolio in reality. This is a limitation of the model: a borrowing composition with a large share in one instrument with a low liquidity constraint means that the issuance algorithm distributes that excess into other instruments by pre-financing, creating a more balanced portfolio than the borrowing composition would dictate. Nevertheless, borrowing compositions constructed in the current way seem to be the best alternative as an input for the optimization algorithm. However, recommending the end-of-simulation portfolios as a goal when creating a medium-term debt management strategy may be a better alternative than setting targets for the borrowing compositions the optimization algorithm returns.

Figure 6: Optimal cost-risk frontier, equal weighted standard deviation, ATR, 1-year share of refinancing



Source: Own calculations

Figure 6 shows the optimal cost-risk frontier with the same, equal weighted risk measures. The current composition of borrowing means that the issuance algorithm tries to keep the initial portfolio composition intact. The corner composition of borrowing show the theoretical costs and risks associated with issuing a single debt instrument throughout the simulation. Corner compositions do not issue the externally given retail debt either. Similar to the optimal composition of borrowing, corner compositions start from the initial portfolio. However, they take neither liquidity constraints nor liquidity costs into account.

Due to all optimal borrowing compositions having a large common share of issuances (that of retail debt), the costs and risks of the corner compositions and the optimal frontier have large differences. Even though - according to the corner compositions - issuing only 15-year fixed bonds would Pareto-dominate every optimal borrowing compositions, this opportunity is unavailable to the debt manager due to both liquidity constraints as well as retail demand.

We can see that retail debt has high expected costs with moderate risks. However, with the forecasts pointing toward an unfavorable exchange rate evolution, retail debt may still prove to be cheaper than FX bonds while having a significantly lower risk level. Figure 6 shows that domestic wholesale debt (both fixed and floating rate) is currently the preferable option for the debt manager if there is enough demand.

8. Conclusions

In this paper, we presented the Hungarian optimal debt portfolio model. We showed how a stochastic simulation model can be used to help the Hungarian Government Debt Management Agency achieve its goal of funding government borrowing needs while keeping costs minimal, and risks at an acceptable level. Our work goes beyond existing stochastic debt management models by using a two-economy Markov regime switching model and a multi-objective optimization framework. We also use a complex estimation of funding liquidity to simulate how supply and demand affect debt management.

The greatest advantage our model offers is flexibility. We can easily add or remove planned instrument types in order to aid the development of new products and estimate their costs and risks. If needed, many of the parameters can be modified to reflect real life changes. The model is well-equipped to handle the relatively complex characteristics of Hungarian government debt. Consequently, it can be adapted or simplified to handle sovereign debt structures with different features. The multi-objective framework allows any debt manager to select risk metrics appropriate to their goals. The relatively low computational requirement is also an advantage of the model, requiring only a single high-end desktop computer for meaningful results in an acceptable timeframe instead of clusters or cloud applications.

The model has some limitations as well, of course, with room for improvement. Without knowledge and expertise in using the model, the results can seem black box at first glance. The calculation of costs also poses an accounting dilemma. Our methodology does not discount future costs when comparing different borrowing compositions. A debt manager might prefer accruing costs at a later time and adjust for inflation or yields. Another limitation is the lack of feedback from interest expenditure or otherwise accrued costs toward the budget deficit or other funding requirements. This compromise was made to lower the computational requirement of the model, and does not need a workaround unless deviations in debt service have a serious budget impact. Finally, in some cases, borrowing compositions do not adequately represent the resulting portfolio. There is possible room for improvement by adjusting the issuance algorithm or including penalty terms if the borrowing composition and the resulting portfolio differ too much.

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